Math 121 4.1 Exponential Functions

Objectives 1) Evaluate and graph exponential equations (review from Math 72)

2) Use the compound interest formulas
   - → compounding m times per year
   - → compounding continuously

   (review from Math 72)

3) Use the present value formula (solve the compound interest formula for P and re-name it)
   - → compounded m times per year
   - → compounded continuously.

4) Compare interest rates

5) Simplify powers of the natural base e

Note: There's no calculus in this section!

e is called the natural base

e, like π, is an irrational number

- it cannot be written as a fraction of two integers
- its decimal does not terminate
- its decimal does not repeat
- e is a constant, always the same value, not a variable.

e 2.718281828 45 90 45 23 53 60 28 74

π 3.1415926 53 58 97 93 23 84 62 64 33

Recall Laws of Exponents

$$x^n \cdot x^m = x^{n+m}$$
$$(x^n)^m = x^{nm}$$
$$x^{-n} = \frac{1}{x^n}$$

Simplify each expression and write as powers of e.

① $\dfrac{e^5 e^{-1}}{e^{-2} e}$

add exponents

$$= \frac{e^{5+(-1)}}{e^{-2+(1)}}$$

$$= \frac{e^4}{e^{-1}}$$

subtract exponents

$$= e^{4-(-1)}$$
$$= \boxed{e^5}$$

② e^0

$$= \boxed{1}$$

③ $\dfrac{1}{e^0} = \dfrac{1}{1} = \boxed{1}$

④ $(1+e^{-x})^2$

$$= (1+e^{-x})(1+e^{-x}) \qquad \text{FOIL}$$
$$= 1 + e^{-x} + e^{-x} + e^{-x} \cdot e^{-x}$$
$$= \boxed{1 + \underbrace{2e^{-x}}_{\text{Combine like terms}} + e^{-2x}} \quad \text{add exponents}$$

⑤ $e^x(1+e^{-x})$

$$= e^x \cdot 1 + e^x \cdot e^{-x} \qquad \text{dist}$$
$$= e^x + e^{x+(-x)} \qquad \text{add exp}$$
$$= e^x + e^0$$
$$= \boxed{e^x + 1}$$

⑥ $e^x(e^x + e^{-x})$

$$= e^x \cdot e^x + e^x \cdot e^{-x} \qquad \text{dist}$$
$$= e^{x+x} + e^{x+(-x)} \qquad \text{add exp}$$
$$= e^{2x} + e^0$$
$$= \boxed{e^{2x} + 1}$$

⑦ $(e^x)^2 + (e^{-x})^2 \qquad \text{mult exp}$

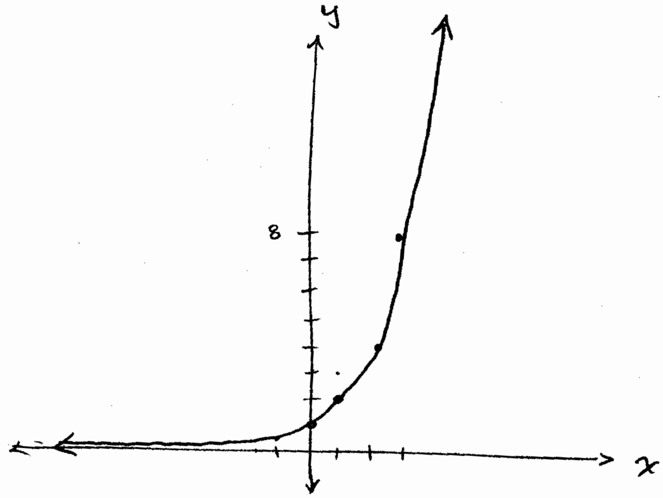
$$= \boxed{e^{2x} + e^{-2x}}$$

⑧ $(e^x + x)(e^{-x} - x)$

$$= e^x \cdot e^{-x} - xe^x + xe^{-x} - x^2 \qquad \text{FOIL}$$
$$= e^0 - xe^x + xe^{-x} - x^2$$
$$= \boxed{1 - xe^x + xe^{-x} - x^2}$$

(9) Graph $f(x) = 2^x$

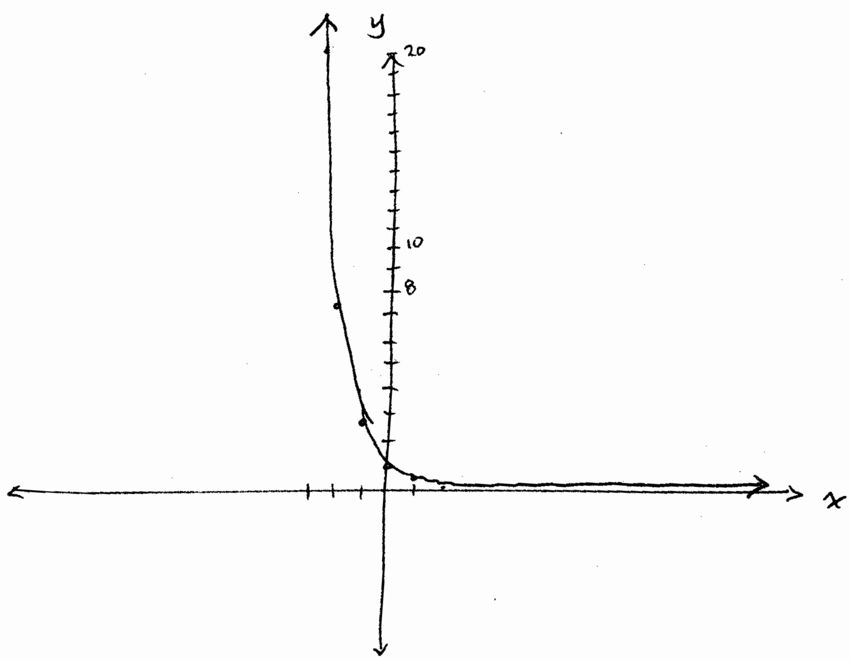
x	$f(x)$
-4	$1/16$
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8
4	16



3 features that are important to graph
1) <u>y-intercept</u>
2) <u>exponential growth</u> to edge of grid
3) <u>horizontal asymptote</u> to edge of grid.

(10) Graph $f(x) = e^{-x}$

x	$f(x)$
-4	$e^4 \approx 54.6$
-3	$e^3 \approx 20.1$
-2	$e^2 \approx 7.4$
-1	$e \approx 2.7$
0	1
1	$1/e \approx .4$
2	$1/e^2 \approx .1$
3	$1/e^3 < .1$
4	$1/e^4 < .1$



An exponential function which decreases as <u>exponential decay</u> instead of exponential growth.

Compound Interest (m compoundings per year)

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

A = amount in account after t years

P = principal (original amount)

r = interest rate, given as %, written as decimal

m = # times compounded per year

t = # years

Interest: The cost of using someone else's money.

- A borrower borrows \$P and pays the lender interest.
- A investor provides \$P to a bank/corporation and receives interest from them.

When interest is compounded, interest is added back to the principal on a regular basis and this accrued interest now earns interest. (Super-wealthy individuals live on the interest payments alone... that's why they needn't work.)

Compound Interest (continuously compounded)

$$A = Pe^{rt}$$

A = amount in account after t years

P = principal

r = interest rate, given as %, written as decimal

t = # years

(11) Find the amount in an account if \$2000 is invested at 3% interest for 5 years

a) compounded quarterly

b) compounded continuously

$A = ?$ $\quad P = \$2000$ $\quad r = .03$ $\quad t = 5$ $\quad$ a) $m = 4$

a) $A = P\left(1+\frac{r}{m}\right)^{mt}$

$= 2000\left(1+\frac{.03}{4}\right)^{4\cdot 5}$

GC with older operating systems must have parentheses (4·5) to do multiply before exponent.

$= 2322.368$

$\approx \boxed{\$2322.37}$

b) $A = Pe^{rt}$

$= 2000e^{(.03)(5)}$

GC with older operating systems will open ( ) automatically.

$= 2323.668$

$\approx \boxed{\$2323.67}$

(12)

a) Your son's fairy godfather will pay all college expenses for your newborn son, provided you will pay \$2000 when your son is 18, and put the necessary money in an account paying 4% compounded monthly for the next 18 years. How much money must you put aside right now?

Present Value Formula (m compoundings)

$$\underset{\text{future value}}{A} = \underset{\text{present value}}{P} \cdot \left(1+\frac{r}{m}\right)^{mt}$$

$$\frac{A}{\left(1+\frac{r}{m}\right)^{mt}} = P$$

Your book confusingly changes meaning of P in the final result ☹

$$\frac{2000}{\left(1+\frac{.04}{12}\right)^{(12\cdot 18)}} = P$$

$\boxed{P = \$974.67}$

b) You bargain with the fairy godfather to get it compounded continuously instead. Now how much do you need?

Present Value Formula (continuous compounding)

$$A = Pe^{rt}$$

$$Ae^{-rt} = \frac{A}{e^{rt}} = P$$

A = future value

P = present value

Again, your book changes the meaning of P!

$$\frac{2000}{e^{(.04)(18)}} = P$$

$$\boxed{P = \$973.50}$$

(13) Which gives a better return?

12% compounded quarterly

or 11.9% compounded continuously

Let $P=1$ and $t=1$.

Compounded Quarterly $A = P\left(1+\frac{r}{m}\right)^{mt}$

$P=1$

$r=.12$

$m=4$

$t=1$

$$= 1\left(1+\frac{.12}{4}\right)^{4\cdot 1}$$

$$\approx 1.12550881$$

Compounded Continuously $A = Pe^{rt}$

$P=1$

$r = 11.9\% = .119$

$t=1$

$$= 1e^{(.119)(1)}$$

$$\approx 1.126369918$$

$$\boxed{\text{11.9\% compounded continuously gives a better return}}$$

Recall Percent Increase? (Math 45, 60, 72)

$$\text{New Amount} = \text{Base} + \% \cdot \text{Base}$$

$$\text{New Amount} = \text{Base}(\underbrace{1+\%}_{\text{subtract 1 to find \%!}})$$

If we view the answers in 13 as percent increases,

12% quarterly yields .12550881 ≈ 12.550881 %

11.9% continuously yields .126369918 ≈ 12.6369918 %

12% and 11.9% are called the <u>nominal rates</u> (in name only) while 12.55% and 12.63% are called any of the following

- effective rate of interest
- annual percentage rate (APR)
- annual percentage yield (APY)